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LETTER TO THE EDITOR

Finite-size scaling at an Ising tricritical point

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Abstract. The tricritical behaviour of a two-dimensional Ising model is investigated by use of a finite-size scaling analysis. The three leading scaling exponents and scaling fields are calculated. The exponents are in excellent agreement with conjectured values and Monte Carlo renormalisation group calculations.

The tricritical behaviour of two-dimensional systems with Ising symmetries has been investigated by a number of authors. The following theoretical techniques have been applied to the problem: Monte Carlo simulations (Landau 1972), ε-expansion renormalisation group (Chang *et al* 1974, Stephen and McCauley 1973), real-space renormalisation group (Neinhuis and Nauenberg 1976), conjectures based on the behaviour of q-state Potts models (Neinhuis *et al* 1979, 1980, den Nijs 1979, Pearson 1980, Neinhuis *et al* 1980), Monte Carlo renormalisation group (Landau and Swendsen 1981), exact solution of a model thought to be in the same universality class as a tricritical Ising model (Baxter 1980, 1981, Huse 1982), and finite-size scaling (Rikvoldt *et al* 1983).

In this letter we report the results of a finite size scaling analysis of an Ising tricritical point. It is similar in approach to that taken by Rikvoldt *et al* but extends their calculation by calculating the three leading scaling exponents and the scaling fields at the tricritical point.

The model used is a two-dimensional Ising model on a square lattice with antiferromagnetic nearest-neighbour bonds and ferromagnetic diagonal-neighbour bonds. The Hamiltonian is

$$\mathscr{H} = J_1 \sum_{(ij)} s_i s_j - J_2 \sum_{\langle ij \rangle} s_i s_j - H \sum_i s_i - H_s \sum_i (-1)^i s_i \tag{1}$$

where $s_i = \pm 1$, (ij) denotes nearest-neighbour pairs and $\langle ij \rangle$ denotes diagonal-neighbour pairs. The fields H and H_s are uniform and staggered magnetic fields respectively. The quantity $(-1)^i$ is unity on one $\sqrt{2}\times\sqrt{2}$ sublattice and -1 on the other. In this work we will take $J_1 = J_2 = J$. This model is exactly equivalent to a lattice gas with nearest-neighbour repulsion and next-nearest-neighbour attraction.

The technique of finite-size scaling is based on the work (Fisher 1971, Fisher and Barber 1972, Barber 1983b) in which the critical behaviour of a large but finite system is considered. This idea was extended (Nightingale 1976) by applying the finite-size scaling hypothesis phenomenologically to systems which are infinite in one direction but finite along all others. The technique has been widely used to find the critical behaviour of two-dimensional systems (Nightingale 1982 and references therein). The key idea is that the correlation length in a system with width L near criticality is given by

$$\xi_L/L \approx Q(L^{y_1}t) \tag{2}$$

where $t=(T-T_c)/T_c$ is the deviation from the critical point and Q is a universal scaling function. The exponent y_1 is the inverse of the correlation length exponent ν. At criticality the correlation lengths should obey

$$\xi_L/L \approx \xi_{L'}/L' \tag{3}$$

for different widths L and L'. The critical temperature is the temperature at which ξ_L scales proportional to the width of the system. Furthermore the exponent y_1 can be determined by

$$y_1 \approx \frac{\ln\left(\frac{\mathrm{d}\xi_L/L}{\mathrm{d}t}\right)\Big|_{T_c}\Big/\left(\frac{\mathrm{d}\xi_{L'}/L'}{\mathrm{d}t}\right)\Big|_{T_c}}{\ln(L/L')}. \tag{4}$$

By using this technique, the critical line in the T, H plane can be determined by sweeping in T at fixed H. The correlation length is determined by the relation

$$\xi = 1/\ln(\lambda_1/|\lambda_2|) \tag{5}$$

where λ_1, λ_2 are the two largest eigenvalues of the Ising transfer marix (Nightingale 1982). Strip widths of up to $L=12$ are used in this calculation. The phase diagram is shown in figure 1. There is a critical line with Ising critical behaviour extending from $(T, H)=(5.263, 0.00)$ to $(2.41, 3.927)$. Along the critical line the thermal exponent is $y_1 \simeq 1.01$ and the magnetic exponent (determined by applying a small staggered field and analysing the L dependence of the correlation length) is $y_2 \simeq 1.87$ in agreement with the exact values of $y_1=1$ and $y_2=1.875$. Along the line from $(T, H)=(2.41, 3.927)$ to $(0.00, 4.00)$ the model exhibits a first-order phase transition from an ordered antiferromagnetic state to the paramagnetic state. This is signalled in a finite-size scaling analysis by the appearance of a third eigenvalue, λ_3, which is asymptotically (in the limit $L\to\infty$) degenerate with λ_1 and λ_2 (Berker and Fisher 1982, Derrida and Herrmann 1983, Cardy and Nightingale 1983). From this near degeneracy

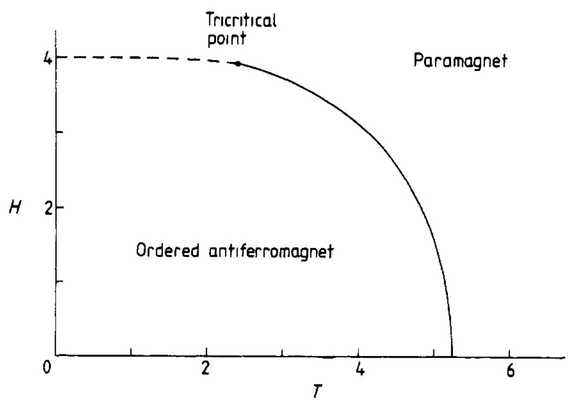

Figure 1. The phase diagram of the Ising antiferromagnet described by equation (1). The full line is a second-order Ising-like transition and the broken line is a first-order transition. The tricritical point is at $T_t=2.41\pm0.02$, $H_t=3.927\pm0.004$.

we can define a persistance length (Rikvoldt *et al* 1982, Derrida and Herrmann 1983, Barber 1983a)

$$\hat{\xi} = 1/\ln(\lambda_1/\lambda_3). \tag{6}$$

The point separating the first- and second-order lines is a tricritical point. At this point the system has three relevant scaling fields which we will denote t, h, h_s. They correspond to temperature, uniform field and staggered field. At the tricritical point the correlation length of a finite-size system should scale like

$$\hat{\xi}_L/L \approx Q(L^{y_1}t, L^{y_2}h_s, L^{y_3}h). \tag{7}$$

The fields t and h lie in the T, H plane and h_s is in the direction of the staggered field. The tricritical point is located by finding the point where both ξ_L and $\hat{\xi}_L$ scale proportional to the width of the system. Figure 2 shows $\hat{\xi}_L/L$ against T along the transition line (which is determined by (3)). The points where $\hat{\xi}_{(L+2)}/(L+2) = \hat{\xi}_L/L$ determine successive estimates of the tricritical temperature T_t. Extrapolations of these estimates give $T_t/J = 2.41 \pm 0.02$ and $H_t/J = 3.927 \pm 0.04$. The critical exponents are determined by using a method which simultaneously determines the scaling directions and exponents (Barber 1983a, Derrida and Herrmann 1983). Equation (4) is applied at the successive estimates of the tricritical point except that the derivatives are taken in all directions, θ, about the tricritical point. The places where the $y(\theta)$'s cross determine the scaling directions and exponents y_1 and y_3. This is shown in figure 3. The t scaling direction is approximately parallel to the H axis and the h scaling direction is tangential to the critical line at the tricritical point. The exponent y_3 can be determined independently by using a formula identical to (4) but replacing ξ_L by $\hat{\xi}_L$ and taking the derivatives parallel to the transition line (Barber 1983a, Derrida and Herrmann 1983). The exponent y_2 is determined by applying a small staggered field and analysing the L dependence of the correlation length. The results obtained

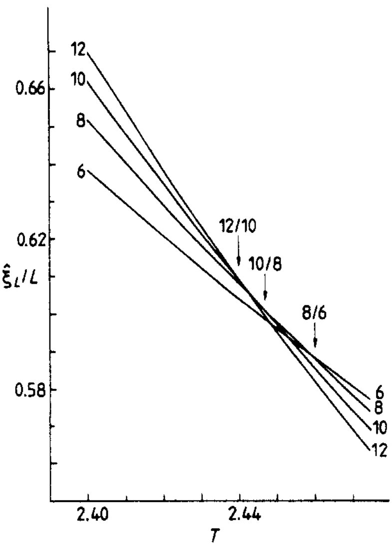

Figure 2. The persistance length on the transition line near the tricritical point. The numbers denote the strip width. The arrows point to successive estimates of the tricritical temperature.

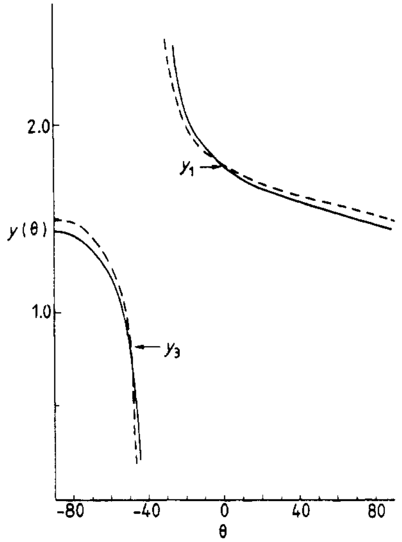

Figure 3. The effective exponent in direction θ (measured in degrees) about the tricritical point. Derivatives were calculated using increments of $\Delta T = 0.005\sin(\theta)$, $\Delta H = 0.001\cos(\theta)$. The full curve is derived from strip widths of 6 and 8. The broken curve is from strip widths of 8 and 10. The two crossing points are estimates of y_1 and y_3. The direction $\theta = 0$ is parallel to the H axis and $\theta = -45$ is approximately parallel to the transition line at the tricritical point.

are presented in table 1 along with the results of previous calculations. The finite-size scaling results are in excellent agreement with the conjectures (Neinhuis *et al* 1979, 1980, den Nijs 1979, Pearson 1980) and the Monte Carlo renormalisation group (Landau and Swendsen 1981). Error estimates for the finite-size scaling results are not given due to the difficulty of extrapolation because of the sizeable corrections to scaling Monte Carlo renormalisation group calculations (Landau 1983) indicate that the leading correction to scaling exponent is approximately −0.3). A rough estimate of 1% to 2% uncertainty is reasonable for y_1 and y_2. The value for the exponent y_3 is somewhat less certain.

Table 1.

Source	y_1	y_2	y_3
a	1.78		1.02
b	1.85		0.65
c	1.800	1.925	0.800
d	1.80 ± 0.02	1.93 ± 0.01	0.84 ± 0.05
e	1.78	1.90	0.77

(a) ε-expansion renormalisation group (Chang *et al* 1974, Stephen and McCauley 1973).
(b) Real-space renormalisation group (Neinhuis and Nauenberg 1976).
(c) Conjecture (Neinhuis *et al* 1979, 1980, den Nijs 1979, Pearson 1980, Baxter 1980, 1981, Huse 1982).
(d) Monte Carlo renormalisation group (Landau and Swendsen 1981).
(e) Finite-size scaling (this work and Rikvoldt *et al* 1983).

In conclusion, the finite-size scaling analysis of a two-dimensional Ising tricritical point gives scaling exponents in excellent agreement with conjectured values and Monte Carlo renormalisation group estimates.

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Note added in proof. A similar calculation has recently been done by Herrmann (1984). His values for y_1 and y_3 agree with ours.

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